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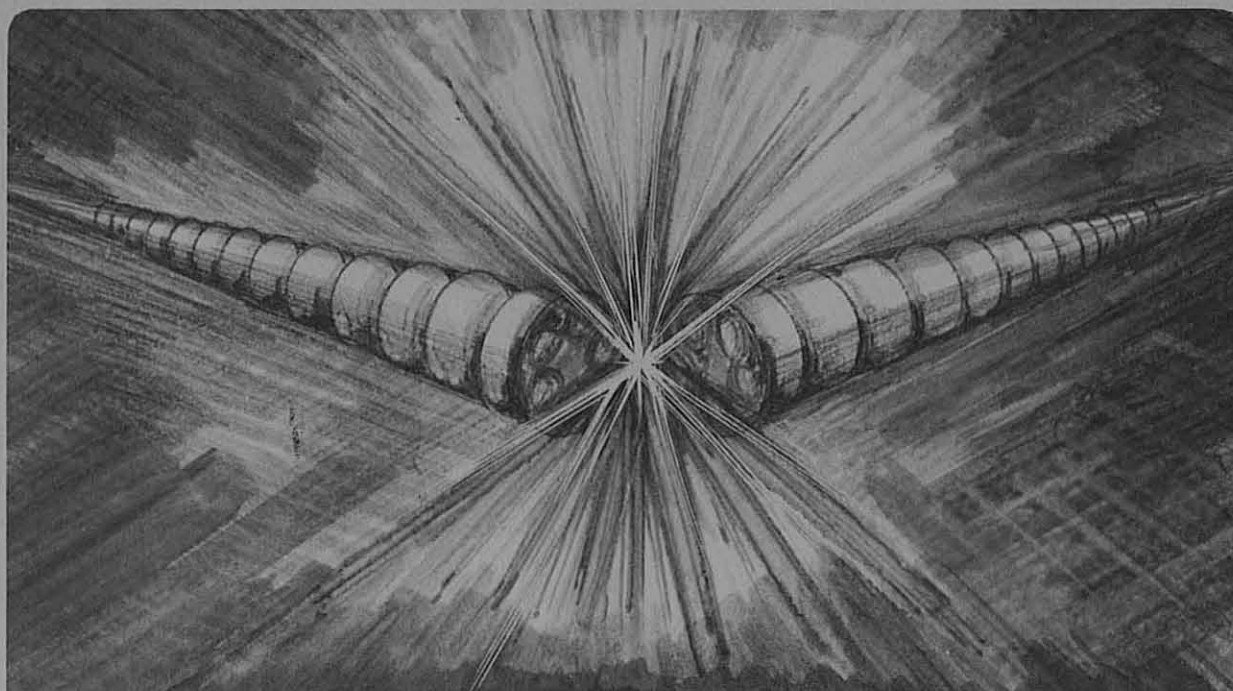## UNIVERSITY OF CALIFORNIA

Accelerator & Fusion Research Division

To be presented at the 7th International Conference on Finite Element Methods in Flow Problems, Huntsville, AL, April 3–7, 1989

Incorporation of a Boundary Condition to Numerical Solution of POISSON's Equation

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October 1988

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ABSTRACT

Two-dimensional and axially-symmetric problems in electrostatics, magnetostatics, or potential fluid flow frequently are solved numerically by means of relaxation techniques -- employing, for example, the finite-difference program POISSON. In many such problems, the "sources" (charges or currents, vorticity, and regions of permeable material) lie exclusively within a finite closed boundary curve and the relaxation process, in principle, then can be confined to the region interior to such a boundary -- provided that a suitable boundary condition is imposed on the solution at the boundary. This paper is a review and illustration of a computational method that uses a boundary condition of such a nature as to avoid the inaccuracies and more extensive meshes present when, alternatively, a simple Dirichlet or Neumann boundary condition is specified on a somewhat more remote outer boundary.

I. INTRODUCTION

Many problems in applied physics and engineering are of such a nature as to require the solution of an apposite partial differential equation in conjunction with pertinent boundary conditions. An effective numerical method often adopted for the solution of such problems is that of "relaxation," in which a finite-difference approximation of the differential equation is solved iteratively on a discretized mesh to obtain an approximate but adequately accurate result conforming to the specified boundary conditions. A frequent application of this technique arises in the solution of problems in electrostatics or magnetostatics, for which one basically requires the solution of an elliptic second-order differential equation (often in just two dimensions) such as that of POISSON for evaluation of a potential function[1].

In the application just cited a convenient boundary condition that can be applied to the discretized problem may be of the Dirichlet type (specification of the function at all boundary points) or of the Neumann type (specification of the normal derivative). In many instances these particular conditions may not be directly applicable, however, and the situation one may wish to specify will instead be that in which no "sources" (i.e., no charges, no currents, no magnetized material, etc.) reside exterior to the mesh boundary. The difficulty that arises under such circumstances can be approximated somewhat inconveniently by use of a quite large mesh, that extends well beyond the region of physical interest (and that, for economy, may employ rather coarse mesh elements in its outer region), so that the condition then applied at the outer boundary becomes of only minor influence.

It would be desirable, however, to develop a more explicit form of boundary condition to characterize the absence of external sources that is accurate and does not require an extension of the mesh. It is our intention to illustrate in the following pages, by reference to two-dimensional electrostatic and magnetostatic problems, how such a boundary condition can be explicitly introduced. It may be hoped that a similar procedure can be seen analogously to have application

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to the solution of some other physical problems governed by linear differential equations. It also will be recognized that, as will be mentioned later in Section II, a simple extension of the procedure will permit in effect the accommodation of remote external sources such as those which will give rise to specified externally applied fields (for example an external field that is uniform at great distances or one with a constant circulation).

II. THE METHOD

The boundary condition we wish to devise is expected not to be strictly of a simple point-to-point nature, but will relate the potential function and its normal derivative in a global manner over the entire boundary. In a discretized array it will be convenient to take the outer boundary of the mesh to be a surface (curve) of <u>constant value</u> for <u>one coordinate</u> of some suitable simple coordinate system and , for the present purposes, also to locate mesh points on a second "interior" nested curve of a similar character. At any reasonable time during the course of the relaxation process, interim values of the potential function as are present at node points along the <u>inner curve</u> than can be employed to generate revised potential values for the <u>outer boundary</u> in a manner consistent with the condition that no external sources are present.

The procedure just described can be used repeatedly in the course of a relaxation solution and can conveniently be applied following one or more complete relaxation passes through the mesh. The newly generated potential values thus assigned to nodes on the outer boundary are then employed as temporarily fixed values for proceeding with a subsequent relaxation cycle.

The generation of revised potential values to be assigned to the outer boundary can be based on fitting potential values on the inner curve to a sum of solutions U_m to the source-free differential equation that individually are each of such a form as could be present in the absence of external sources. This sum can then be evaluated at node points on the outer boundary to provide the revised values required for proceeding with the relaxation. Thus, in the event that cylindrical polar coordinates are employed in a two-dimensional problem, suitable functions U_m to serve as solutions to $\nabla^2 U = 0$ would be of the form $r^m \cos m\theta$ and $r^m \sin m\theta$ with m a <u>positive</u> integer. [An additional term of the form $-\lambda \ln r^2$ also will be required in the event that a non-zero net charge is present in the interior and so implies the occurrence of a corresponding "opposite charge at infinity." An adjustable constant term similarly may be required to fix the potential in a unique way without violating other specifications of the problem.]

The algorithm to generate revised values of potential at points on the outer boundary will be a linear one, characterized by a "transfer matrix" whose elements will be fixed once the geometry of the problem is specified. In practice, it may be sufficient (and prudent) to confine the number of functions U_m that are employed in this process to a value (e.g., 30 or 50) somewhat smaller than the number of points on the inner curve whose potential values V are to be so represented. In this case the elements of the transfer matrix may be evaluated to represent a least-squares fit of the function V to values given at the nodal points of the inner curve. The functions U_m of the development upon which determination of these matrix elements will be based of course should include only those with the symmetry of the problem to be solved.

We present later in this paper some specific details concerning coordinate systems that we have found useful in applying the aforementioned techniques to solution of electrostatic and magnetostatic problems through use of the program POISSON. Additional details, and several specific examples, have been presented earlier in LBL and conference reports[2]. Our use of the transfer matrix to revise potential values on the outer boundary has normally been at the completion of each full relaxation pass and without application of any over-relaxation (or under-relaxation) parameter in connection with this transfer operation. In problems in which external sources <u>are present</u>, so as to provide a <u>known externally-applied</u> field or potential function in the region of interest, the procedures described here may still be employed (method of "superposition"[2f, h]) if such supplemental potentials are first subtracted from the potentials on the inner boundary and then

2

are reevaluated and added to potentials found at the outer boundary subsequent to the transfer operation. The details of these procedures, and other possibly more general issues relating to extensions of the work described here, clearly would benefit from imaginative examination by experienced numerical analysts.

III. THE PROGRAM POISSON

Two-dimensional programs of the type represented by TRIM and POISSON treat electrostatic and magnetostatic problems in a similar manner. These programs basically comprise (i) a mesh generator, to provide a mesh formed of irregular triangles that can be arranged to conform to irregular surfaces and boundaries, (ii) a relaxation routine for iterative solution of the finite-difference equations, and (iii) an editor to analyze and present the results of the computation. The differential equation to be represented in Cartesian cases is of the form

$$\nabla \cdot (\gamma \nabla A) + S = 0 ,$$

where the working variable A will represent either the scalar electrostatic potential function or the (z-directed) component of a magnetic vector potential A_z, γ denotes the electric susceptibility or the reciprocal of the magnetic permeability (subject to repeated revision if given as a function of field strength), and S is a measure of the strength of prescribed interior sources (charge or current, per unit area). In regions free of sources and of permeable material, the controlling equation then reduces simply to the homogeneous Laplace equation.

In cases with cylindrical symmetry (problems free of azimuthal dependence, for which ρ, z could serve as suitable cylindrical coordinates) the relaxation computations can be organized in a similar way if one adopts as a working variable the quantity $A^* = \rho A_\phi$ in the magnetostatic case (and make a corresponding replacement $\gamma = (\rho\mu)^{-1}$ with respect to permeability). Thus, in regions devoid of sources (S = 0 and μ constant), the homogeneous equations for the magnetic potentials A_z or A^* assume the following respective forms in the 2-D (Cartesian) and cylindrically-symmetric cases to represent $\nabla \times \left[\frac{1}{\mu} \nabla \times \vec{A} \right] = 0$;

Cartesian (x, y mesh):

$$\frac{\partial^2 A_z}{\partial x^2} + \frac{\partial^2 A_z}{\partial y^2} = 0 ;$$

Cylindrically-Symmetric (ρ, z cylindrical-coordinate mesh):

$$\frac{\partial}{\partial \rho} \left[\frac{1}{\rho} \frac{\partial A^*}{\partial \rho} \right] + \frac{\partial}{\partial z} \left[\frac{1}{\rho} \frac{\partial A^*}{\partial z} \right] = 0 .$$

In designing a mesh to accommodate electrostatic or magnetostatic problems in which one can specify the complete absence of sources beyond a certain region, it is convenient to employ as mesh boundaries curves that correspond to constant values of one coordinate of a coordinate system that can readily be employed to enclose efficiently the region of physical interest. Solutions of Laplace's equation that are expressed in terms of the coordinates of such a coordinate system then can be used to guide the development of an algorithm to apply the method outlined in Section II at the boundary of the mesh and thereby take recognition of the absence of sources exterior to this boundary. In Table I we indicate the coordinate systems and the corresponding types of terms that we have used to describe functions A_z or A^* in the boundary regions of such magnetostatic problems for various configurations. The specific application in such configurations to a version of the program POISSON is described in further detail in a sequence of reports[2] from the Lawrence Berkeley Laboratory.

TABLE I

Type of Boundary	Coordinates (η, ξ) Employed in Boundary Region[a]	Terms of Series to be transferred to outer Boundary [b]
Straight 2D:		For A_z
Cylindrical Polar	$\eta = \rho = (x^2 + y^2)^{1/2}$, $\xi = \theta \quad \tan^{-1}(y/x)$	$\eta^{-m} \dfrac{\cos}{\sin}(m\xi)$
Elliptic	$c \cosh(\eta + i\xi) = x + iy$	$e^{-m\eta} \dfrac{\cos}{\sin}(m\xi)$
Cylindrically Symmetrical:		For $A^* = \rho A_\phi$
Spherical	$\eta = r = (\rho^2 + z^2)^{1/2}$, $\xi = \tan^{-1}(\rho/z)$	$\eta^{-m} \cdot \sin\xi \, P_m^1(\cos\xi)$
Prolate Ellipsoidal	$c\left[(\eta^2 - 1)(1 - \xi^2)\right]^{1/2} = \rho$, $c\eta\xi = z$	$\left[(\eta^2 - 1)(-Q'_m(\eta))\right] \cdot \left[(1 - \xi^2)^{1/2}(P_m^1(\xi))\right]$
Toroidal	$\dfrac{a \sinh\eta}{\cosh\eta - \cos\xi} = \rho$, $\dfrac{a \sin\xi}{\cosh\eta - \cos\xi} = z$	$\dfrac{\sinh\eta}{(\cosh\eta - \cos\xi)^{1/2}} = P_{m-1/2}^1(\cosh\eta) \dfrac{\cos}{\sin}(m\xi)$

(a) The coordinate η will assume the respective constant values η_{inner} and η_{outer} on the inner and outer boundaries of the mesh.

(b) The functions P_m^1 denote associated Legendre functions of order 1, Q'_m represents the derivative of the ordinary function Q_m.

4

IV. CONCLUSION

We illustrate the utility of the methods described in this paper by computational results, shown in Fig. 1 - 5, that were obtained from incorporation of the proposed boundary conditions into a POISSON program run on a Hewlett-Packard A900 computer. We have found such computations helpful in the analysis of SSC (Superconducting Super Collider) superconducting dipole magnets.

V. REFERENCES

1. Winslow, A. M., "Numerical Solution of the Quasilinear POISSON Equation in a Nonuniform Triangular Mesh," J. Comput. Physics, v. 1 (No. 2), 149-172 (November 1966).
2. Previous applications of a boundary condition similar to those introduced here have been described, for various coordinate systems, in the following reports of the Lawrence Berkeley Laboratory: (a) ESCAR-28 (1975), (b) LBID-172 (1980), (c) LBL-17064 (1984), (d) LBL-18063 (1984), (e) LBL-18798 (1984), (f) LBL-19050 (1985), (g) LBL-19483 (1985), (h) LBL-19172 (1985), (i) LBL-20893 (1986), (j) LBL-22985 (1987).
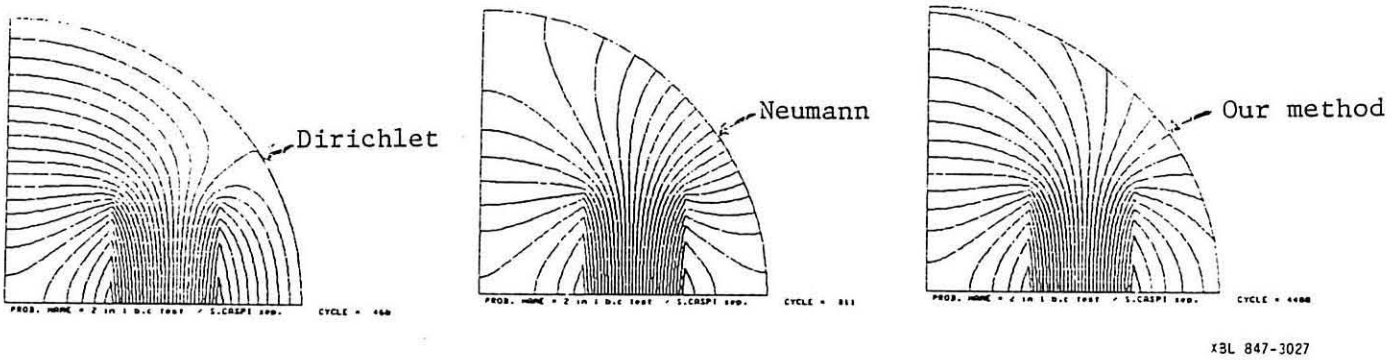
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Fig. 1 Flux lines developed by two 2-D current sheets, to form a "two-dimensional solenoid." The figures contrast the field lines incorrectly obtained through the use of Dirichlet or Neumann boundary conditions (at a circular mesh boundary) with the pattern resulting from application of the method described in this paper.[2c] Numerical checks applied to this problem indicate the correctness of the results obtained by this latter method.
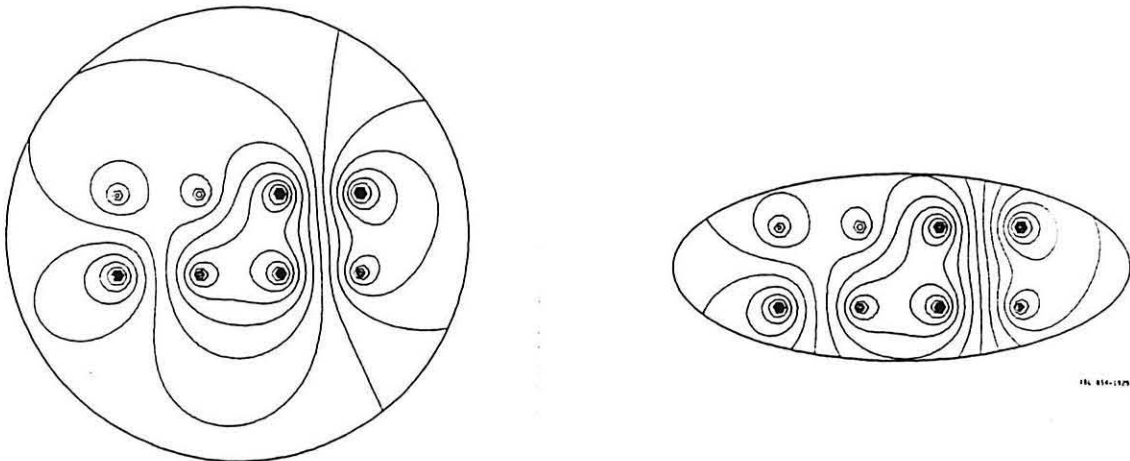

Fig. 2 Flux lines about unequally excited current filaments in a 2-D geometry, illustrting the economy of mesh area (and associated convergence rate) that can be obtained by use of a suitably proportioned elliptical boundary in preference to a circular boundary.[2e] Numerical checks indicate that use of these different boundaries leads to identical results that are in good agreement with direct numerical calculations from Ampere's Law.
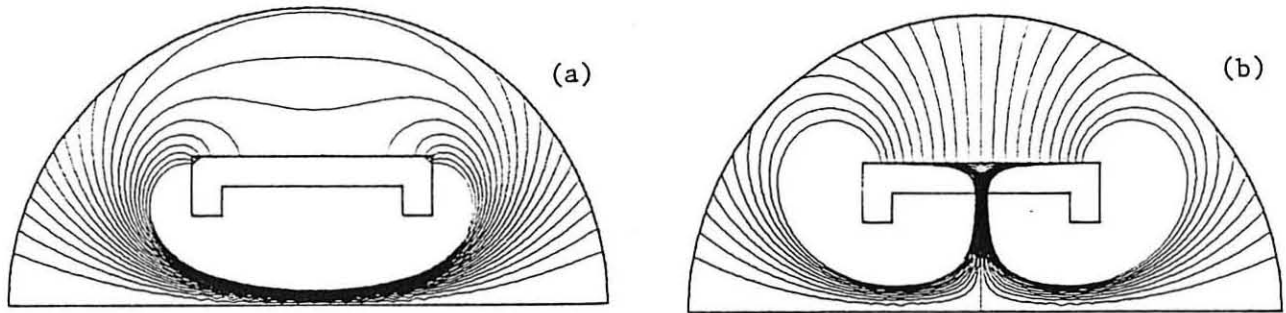
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Fig. 3 Field-line diagrams obtained for a solenoidal structure (rotational symmetry about the lower edge of the diagrams) for (a) excitation even with respect to the midplane and (b) excitation odd about the midplane.[2]
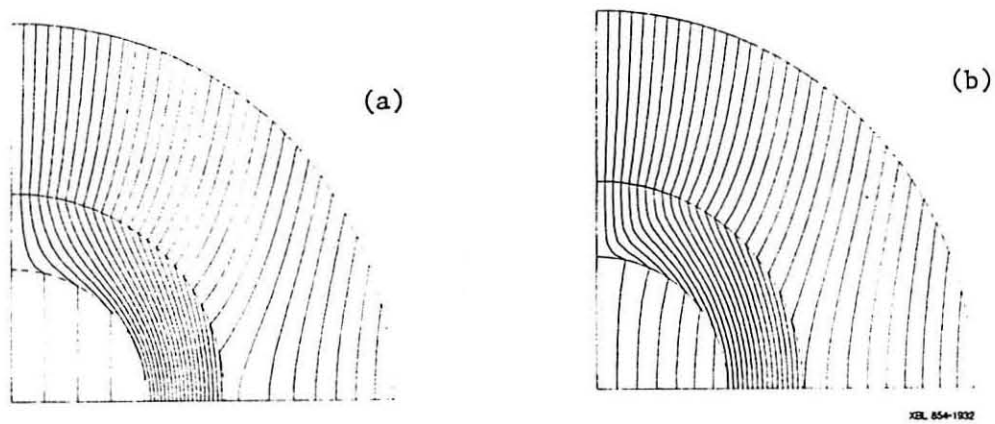

Fig. 4. Illustrations of field patterns that develop from imposition of a uniform externally applied field about a cylindrical (2-D) iron ring: (a) for a ring of constant permeability, $\mu = 10$; (b) for a realistic field-dependent permeability.[2h]
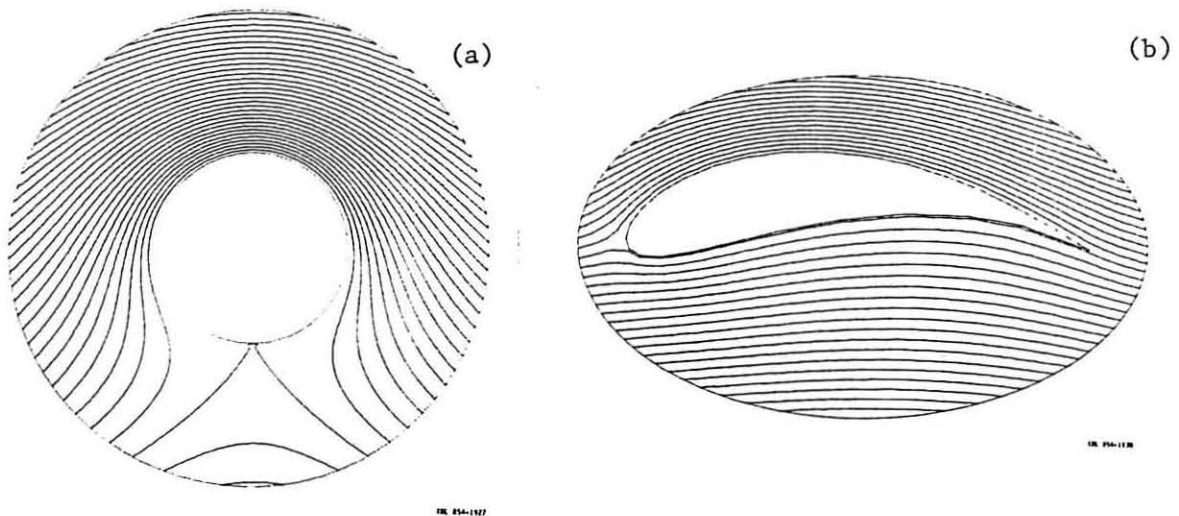

Fig. 5. Illustrations of the superposition of an externally applied field (with circulation) in a straight 2-D problem: (a) Uniform flow over a circular cylinder, with the strength of the applied circulation adjusted to produce a single stagnation point; (b) similarly adjusted flow about a Joukowski airfoil.[2f]